Poisson boundary of a relativistic diffusion in spatially flat and fast expanding Robertson-Walker space-times

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Summary. We determine the long-time asymptotic behavior of a relativistic diffusion taking values in the unitary tangent bundle of a spatially flat and fast expanding Robertson-Walker space-time. We prove in particular that the Poisson boundary of the diffusion can be identified with the causal boundary of the manifold.

Key words: Brownian motion, relativistic diffusion, Lorentzian manifolds, Poisson boundary, causal boundary.

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1 Introduction

The study of Brownian motion on a Riemannian manifold shows that the short-time and long-time asymptotic behavior of the process strongly reflects the geometry of the underlying manifold. Considering the importance of heat kernels in Riemannian geometry, it appears very natural to investigate the links between geometry and asymptotics of Brownian paths in a Lorentzian setting. In his seminal work [Dud66], R.M. Dudley showed that a relativistic diffusion, *i.e.* a diffusion process with values in a Lorentz manifold whose law is Lorentz-covariant, cannot exist in the base space, but makes sense at the level of the tangent bundle. More precisely, Dudley showed that there is no Lorentz-covariant diffusion in the Minkowski space-time but that there exists a unique Lorentz-covariant diffusion with values in its (pseudo)-unitary tangent bundle. This process, that we will name *Dudley's diffusion* in the sequel, is simply obtained by integrating the classical hyperbolic Brownian motion on the unitary tangent space.

In [FLJ07], J. Franchi and Y. Le Jan extended Dudley's construction to the realm of general relativity by defining, on the future-directed half of the unitary tangent bundle $T^1\mathcal{M}$ of an arbitrary Lorentz manifold $\mathcal{M}$, a diffusion which is Lorentz-covariant. This process, that we will simply call *relativistic diffusion*, is the Lorentzian analogue of the classical Brownian motion on a Riemannian manifold. It can be seen either as a random perturbation of the timelike geodesic flow on the unitary tangent bundle, or as a stochastic development of Dudley's diffusion in a fixed tangent space, see Sect. 3 of [FLJ07]. In [FLJ11], Franchi and Le Jan then generalized their original construction by introducing the so-called "curvature diffusions", whose quadratic variation is allowed to depend locally on the curvature of the underlying space-time.

In the case when the underlying manifold is the Minkowski space-time, the long-time asymptotics of the above relativistic diffusion, which coincides with Dudley's original one, is well understood. It was first studied by Dudley himself in [Dud66, Dud73] where it is shown that the process is transient, and escapes to infinity in a random preferred direction, see also [FLJ07] Sect. 2 for a simple proof. In [Bai08], I. Bailleul performed the full determination of the Poisson boundary of the relativistic diffusion, *i.e.* the set of bounded harmonic functions on Minkowski phase space endowed with the differential operator which is the infinitesimal generator of the diffusion. Recall that this is equivalent to the determination of the invariant σ-field of the relativistic diffusion. Moreover, Bailleul gave a geometric description of the Poisson boundary of the relativistic diffusion which can be formulated in terms of the causal boundary of Minkowski space-time. Finally in [Tar13], C. Tardif completed the picture by computing the Lyapunov spectrum and stable manifolds of the stochastic flow associated to the lift of Dudley's diffusion on the Poincaré group.

As for the usual Brownian motion on a general Riemannian manifold, there is no hope to fully determinate the asymptotic behavior of the relativistic diffusion on an arbitrary Lorentzian manifold: it could depend heavily on the base space, see e.g. [ATU09] and its references in the case of Cartan–Hadamard manifolds. In fact, the difficulty is a priori greater in the Lorentzian context: first because of the non-positivity of the underlying metric, then because the relativistic diffusion does not live on the base manifold, but on its pseudo-unit tangent bundle, so that it is basically seven-dimensional when the base manifold have four dimensions, and there is no general reason that it must contain one or more lower-dimensional sub-diffusions. On the contrary, recall that in the case of a constantly curved Riemannian manifold, the Brownian motion fortunately always admits a one-dimensional sub-diffusion: the radial sub-diffusion.

Nevertheless, the study of the relativistic diffusion has been led in details in some significant examples of Lorentzian manifolds. Thereby, in [FLJ07] and [Fra09], the authors studied the long-time behavior of the diffusion in Schwarzschild-Kruskal-Szekeres space-time and Gödel space-time respectively. Although they did not reach the full determination of the Poisson boundary, they achieved to describe the almost sure asymptotics of diffusion's paths and came up with the conclusion that they asymptotically behave like random light-like geodesics.

Recently in [Ang13], we studied in details the long-time asymptotic behavior of the relativistic diffusion in the case when the underlying space-time belong to a large class of Lorentz manifold: Robertson-Walker space-times, see Sect. 2.1. Our study confirm [FLJ07]'s predictions concerning the links between the diffusion's paths and light-like geodesics. We show in particular that the relativistic diffusion's paths converge almost surely to random points of the causal boundary $\partial\mathcal{M}_c^+$ [GKP72, AnF07] of the base manifold $\mathcal{M}$.

Theorem (Theorem 3.1 in [Ang13]). *Let $\mathcal{M} := (0,T) \times_\alpha M$ be a Robertson-Walker space-time. Let $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$ and let $(\xi_s, \dot{\xi}_s)_{0\leq s\leq \tau}$ be the relativistic diffusion in $T^1_+\mathcal{M}$ starting from $(\xi_0, \dot{\xi}_0)$. Then, almost surely as s goes to the explosion time τ of the diffusion, the first projection ξ_s converges to a random point ξ_∞ of the causal boundary $\partial\mathcal{M}_c^+$.*

The purpose of this paper is to push the analysis further by showing that, in the case of a spatially flat and fast expanding Robertson-Walker space-time $\mathcal{M} = (0,+\infty) \times_\alpha \mathbb{R}^3$, the Poisson boundary of the diffusion is precisely generated by the single random variable ξ_∞ of the causal boundary $\partial\mathcal{M}_c^+$, which in that case can be identified with a spacelike copy of the Euclidian space $\mathbb{R}^3$ (see Theorem 4.3 of [AnF07]). Namely, we prove the following result:

Theorem (Theorems 1 and 2 below). *Let $\mathcal{M} := (0,+\infty) \times_\alpha \mathbb{R}^3$ be a Robertson-Walker space-time where α has exponential growth. Let $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$ and let $(\xi_s, \dot{\xi}_s)_{s\geq 0} = (t_s, x_s, \dot{t}_s, \dot{x}_s)_{s\geq 0}$ be the relativistic diffusion in $T^1_+\mathcal{M}$ starting from $(\xi_0, \dot{\xi}_0)$. Then, almost surely as s goes to infinity, the spatial projection x_s converges to a random point x_∞ in $\mathbb{R}^3$, and the invariant sigma field of the whole diffusion $(\xi_s, \dot{\xi}_s)_{s\geq 0}$ coincides almost surely with $\sigma(x_\infty)$.*

The above result is the first computation of the Poisson boundary of the relativistic diffusion in the case of a curved manifold, where classical Lie group methods do not apply. It can be seen as a complementary result of those of [Bai08] in the flat case of Minkowski space-time.

The article is organized as follows. In the next section, we briefly recall the geometrical background on Robertson-Walker space-times and the definition of the relativistic diffusion in this setting. In Section 3, we then state the results concerning the asymptotic behavior of the relativistic diffusion and its Poisson boundary. The fourth section is dedicated to the proofs of these results.

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2 Geometrical and probabilistic background

The Lorentz manifolds we consider here are Robertson-Walker space-times. These manifolds are named after H. P. Robertson and A. G. Walker [Rob35, Wal37] and their work on solutions of Einstein's equations satisfying the "cosmological principle". They are the geometric framework to formulate the theory of Big-Bang in General Relativity.

2.1 Robertson-Walker spacetimes

The constraint that a space-time satisfies both Einstein's equations and the cosmological principle implies it has a warped product structure, see e.g. [Wei72] p. 395–404. A Robertson-Walker space-time, classically denoted by $\mathcal{M} := I \times_\alpha M$, is thus defined as a Cartesian product of a open interval $(I, -dt^2)$ (the base) and a Riemannian manifold (M, h) of constant curvature (the fiber), endowed with a Lorentz metric of the following form $g := -dt^2 + \alpha^2(t)h$, where α is a positive function on I, called the *expansion function.* Classical examples of Robertson-Walker space-times are the (half)$-$Minkowski space-time, Einstein static universe, de Sitter and anti-de Sitter space-times etc.

The geometry of Robertson-Walker space-times is both simple in the sense that one can for example explicitly integrate the geodesic equations, but it is rich because of the latitude allowed in the choice of the fiber, the base, and the expansion function. A detailed study of the relativistic diffusion in a general Robertson-Walker space-time has been led in [Ang13] where we characterized the almost-sure long-time behavior of the diffusion. We focus here on the case when the interval I is unbounded and the fiber is Euclidian. Namely, we consider Robertson-Walker space-times $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$, where α satisfy the following hypotheses:

Hypotheses.

1. The function α is of class C^2 on $(0, +\infty)$ and it is increasing and log$-$concave, *i.e.* the Hubble function $H := \alpha'/\alpha$ is non-increasing.
2. The expansion function α has exponential growth, *i.e.* the limit $H_\infty := \lim_{t \to +\infty} H(t)$ is positive.

Among the spaces frequently used in cosmology, the de Sitter space is an example of Robertson-Walker space-time with exponential growth.

Remark 1. *The hypothesis of* log$-$*concavity of the expansion function is classical, it appears natural from both physical and mathematical points of view, see e.g. [HE73, AC07].*

A manifold $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$ is naturally endowed with a global chart $\xi = (t, x)$ where $x = (x^1, x^2, x^3)$ are the canonical coordinates in $\mathbb{R}^3$. At a point (t, x), the scalar curvature of such Robertson-Walker space-time is $R = -6(\alpha''(t)/\alpha(t) + \alpha'^2(t)/\alpha^2(t))$, in particular, the manifolds we consider are not flat in general. In the case of a "true" exponential expansion, that is when $\alpha(t) = \exp(H \times t)$ for a positive constant H, the Robertson-Walker space-time $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$ is an Einstein manifold, *i.e.* its Ricci tensor is proportional to its metric.

2.2 The relativistic diffusion in Robertson-Walker spacetimes

The sample paths $(\xi_s, \dot{\xi}_s)$ of the relativistic diffusion introduced in [FLJ07] are time-like curves that are future-directed and parametrized by the arc length s so that the diffusion actually live on the positive part of the unitary tangent bundle of the manifold, that we denote by $T^1_+\mathcal{M}$. The infinitesimal generator of the diffusion is the following hypoelliptic operator

$$\mathcal{L} := \mathcal{L}_0 + \frac{\sigma^2}{2}\Delta_{\mathcal{V}},$$

where $\mathcal{L}_0$ generates the geodesic flow on $T^1\mathcal{M}$, $\Delta_{\mathcal{V}}$ is the vertical Laplacian, and σ is a real parameter. Equivalently, if ξ^μ is a local chart on $\mathcal{M}$ and if $\Gamma^\mu_{\nu\rho}$ are the usual Christoffel symbols, the relativistic diffusion is the solution of the following system of stochastic differential equations, for $0 \leq \mu \leq d = \dim(\mathcal{M})$:

$$\left\{ \begin{array}{l} d\xi^\mu_s = \dot{\xi}^\mu_s ds, \\ d\dot{\xi}^\mu_s = -\Gamma^\mu_{\nu\rho}(\xi_s)\, \dot{\xi}^\nu_s \dot{\xi}^\rho_s ds + d \times \dfrac{\sigma^2}{2}\, \dot{\xi}^\mu_s ds + \sigma dM^\mu_s, \end{array} \right. \tag{1}$$

where the brakets of the martingales M^μ_s are given by

$$\langle dM^\mu_s,\, dM^\nu_s \rangle = (\dot{\xi}^\mu_s \dot{\xi}^\nu_s + g^{\mu\nu}(\xi_s))ds.$$

In the case of a manifold $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$ endowed with its natural global chart, the metric is $g_{\mu\nu} = \mathrm{diag}(-1, \alpha^2(t), \alpha^2(t), \alpha^2(t))$, and the only non vanishing Christoffel symbols are $\Gamma^0_{i\,i} = \alpha(t)\alpha'(t)$, and $\Gamma^i_{0\,i} = H(t)$ for $i = 1, 2, 3$. Thus, in the case of a spatially flat Robertson-walker space-time, the system of stochastic differential equations (1) that defines the relativistic diffusion simply reads:

$$\left\{ \begin{array}{ll} dt_s = \dot{t}_s ds, & d\dot{t}_s = -\alpha(t_s)\, \alpha'(t_s) |\dot{x}_s|^2 ds + \dfrac{3\sigma^2}{2} \dot{t}_s ds + dM^{\dot{t}}_s, \\ \\ dx^i_s = \dot{x}^i_s ds, & d\dot{x}^i_s = \left(-2H(t_s)\dot{t}_s + \dfrac{3\sigma^2}{2} \right) \dot{x}^i_s\, ds + dM^{\dot{x}^i}_s, \end{array} \right. \tag{2}$$

where $|\dot{x}_s|$ denote the usual Euclidian norm of $\dot{x}_s$ in $\mathbb{R}^3$ and

$$\left\{ \begin{array}{l} d\langle M^{\dot{t}},\, M^{\dot{t}} \rangle_s = \sigma^2 \left(\dot{t}^2_s - 1 \right) ds, \quad d\langle M^{\dot{t}},\, M^{\dot{x}^i} \rangle_s = \sigma^2\, \dot{t}_s \dot{x}^i_s ds, \\ \\ d\langle M^{\dot{x}^i},\, M^{\dot{x}^j} \rangle_s = \sigma^2 \left(\dot{x}^i_s \dot{x}^j_s + \dfrac{\delta_{ij}}{\alpha^2(t_s)} \right) ds. \end{array} \right.$$

Moreover, the parameter s being the arc length, we have the pseudo-norm relation:

$$\dot{t}^2_s - 1 = \alpha^2(t_s) \times |\dot{x}_s|^2. \tag{3}$$

Remark 2. *The sample paths being future-directed, from the above pseudo-norm relation, we have obviously $\dot{t}_s \geq 1$, in particular as long as it is well defined, the process t_s is a strictly increasing and $t_s > s$.*

3 Statement of the results

We can now state our results concerning the asymptotic behavior of the relativistic diffusion and its Poisson boundary in a spatially flat and fast expanding Robertson-Walker space-time. For the sake of clarity, the proofs of these different results are postponed in Section 4. For the whole section, let us thus fix a spatially flat Robertson-Walker space-time $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$, where α satisfies the hypotheses stated in Sect. 2.1.

3.1 Existence, uniqueness, reduction of the dimension

Naturally, the first thing to do is to ensure that the system of stochastic differential equations (2) admits a solution, and possibly to exhibit lower dimensional sub-diffusions that will facilitate its study. This is the object of the following proposition.

Proposition 1. *For any $(\xi_0, \dot{\xi}_0) = (t_0, x_0, \dot{t}_0, \dot{x}_0) \in T^1_+\mathcal{M}$, the system of stochastic differential equations (2) admits a unique strong solution $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ starting from $(\xi_0, \dot{\xi}_0)$, which is well defined for all positive proper times s. Moreover, this solution admits the two following sub-diffusions of dimension two and four respectively:*

$$(t_s, \dot{t}_s)_{s\geq 0}, \qquad (t_s, \dot{t}_s, \dot{x}_s/|\dot{x}_s|)_{s\geq 0}.$$

Remark 3. *Given a point $(\xi, \dot{\xi}) \in T^1_+\mathcal{M}$, we will denote by $\mathbb{P}_{(\xi,\dot{\xi})}$ the law of the relativistic diffusion starting from $(\xi, \dot{\xi})$ and by $\mathbb{E}_{(\xi,\dot{\xi})}$ the associated expectation. Unless otherwise stated, the word "almost surely" will mean $\mathbb{P}_{(\xi,\dot{\xi})}-$almost surely. The two above sub-diffusions will be called the* temporal *and* spherical *diffusions respectively.*

Owing to Proposition 1, in Sect. 4.2, we shall determine the asymptotic behavior of the seven-dimensional relativistic diffusion $(\xi_s, \dot{\xi}_s)$ gradually: we first determine the asymptotic behavior of the temporal sub-diffusion, then of the spherical sub-diffusion, and finally deduce the asymptotic behavior of the whole diffusion. These asymptotic results are summarized in the next paragraph.

3.2 Asymptotics of the relativistic diffusion

As conjectured in [FLJ07], we show that the relativistic diffusion asymptotically behaves like light rays, *i.e.* light-like geodesics. Indeed, from Remark 2 we know that the first projection t_s of the (non-Markovian) process $\xi_s = (t_s, x_s) \in \mathcal{M}$ goes almost-surely to infinity with s. We shall prove that its spatial part x_s converges almost surely to a random point x_∞ in $\mathbb{R}^3$, so that the diffusion asymptotically follows a line D_∞ in $\mathcal{M}$, see figure 3.2 below, which is the typical behavior of a light-like geodesic. Moreover, we shall see that the normalized derivative $\dot{x}_s/|\dot{x}_s|$ is recurrent, *i.e.* the curve $(\xi_s)_{s\geq 0}$ actually winds along the line D_∞ in a reccurent way.

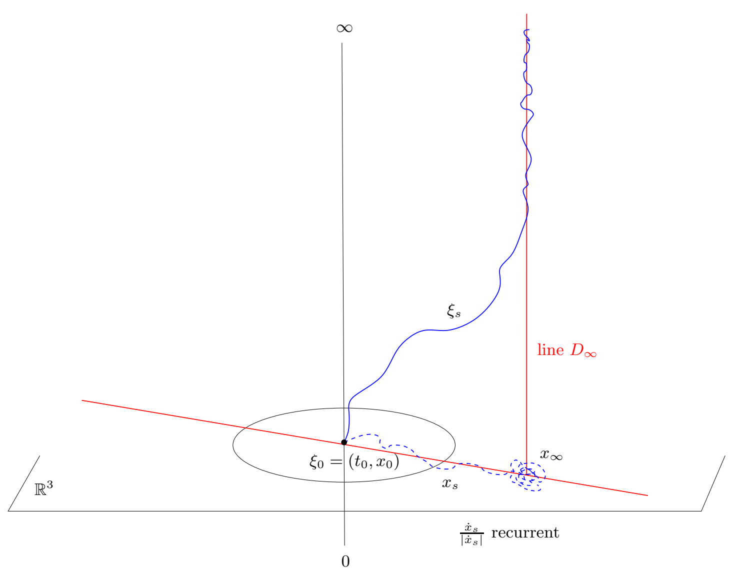

Figure 1: Typical path of the relativistic diffusion in $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$.

To state precise results, let us introduce the following notations. Given two positive constants a and b, let $\nu_{a,b}$ be the probability measure on $(1, +\infty)$ admitting the following density with respect to Lebesgue measure:

$$\nu_{a,b}(x) := C_{a,b} \times \sqrt{x^2 - 1} \times \exp\left(-\frac{2a}{b^2}x\right),$$

where $C_{a,b}$ is the normalizing constant. If f is a $\nu_{a,b}-$integrable function, we will write

$$\nu_{a,b}(f) := \int f(x)\nu_{a,b}(x)dx.$$

The following theorem summarizes the almost sure asymptotics of the relativistic diffusion, its proofs is given in Sect. 4.2 below.

Theorem 1. *Let $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$, and let $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ be the relativistic diffusion starting from $(\xi_0, \dot{\xi}_0)$. Then as s goes to infinity, we have the following almost sure asymptotics:*

1. *the non-Markovian process $\dot{t}_s$ is Harris-recurrent in $(1, +\infty)$. Moreover, if f is a monotone, $\nu_{H_\infty,\sigma}-$integrable function, or if it is bounded and continuous, one has the almost sure convergence:*

$$\lim_{s\to+\infty} \frac{1}{s} \int_0^s f(\dot{t}_u)du = \nu_{H_\infty,\sigma}(f).$$

   In particular, one has $t_s/s \xrightarrow{a.s.} \nu_{H_\infty,\sigma}(\mathrm{Id}) > 0$ when s goes to infinity.

2. *the spatial projection x_s converges almost surely to a random point x_∞ in $\mathbb{R}^3$.*

3. *the normalized spatial derivative $\dot{x}_s/|\dot{x}_s|$ is a time-changed Brownian motion on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, in particular it is recurrent.*

The above asymptotic results can be rephrased concisely thanks to the notion of causal boundary introduced in [GKP72]. In fact, in a spatially flat and fast expanding Robertson-Walker space-time, the causal boundary identifies with a spacelike copy of $\mathbb{R}^3$, a causal curve $\xi_u = (t_u, x_u)$ converging to a point ξ_∞ of the boundary iff $t_u \to +\infty$ and $x_u \to x_\infty \in \mathbb{R}^3$, see [AnF07]. As noticed in the introduction, the convergence of the process ξ_s to a random point of the causal boundary generalizes to a general Robertson-Walker space-time, see [Ang13].

3.3 Poisson boundary of the relativistic diffusion

We now describe the Poisson boundary of the relativistic diffusion, that is we determine its invariant sigma field $\mathrm{Inv}((\xi_s, \dot{\xi}_s)_{s\geq 0})$ or equivalently the set of bounded harmonic functions with respect to its infinitesimal generator $\mathcal{L}$. Owing to Theorem 1, the processes $\dot{t}_s$ and $\dot{x}_s/|\dot{x}_s|$ being recurrent, it is tempting to assert that the only non trivial asymptotic variable associated to the relativistic diffusion is the random point $x_\infty \in \mathbb{R}^3$. Indeed, using coupling techniques, we first prove in Sect. 4.3 (Propositions 4 and 5) that the invariant sigma fields of the temporal and spherical sub-diffusions are trivial. Finally, thanks to an extra argument that takes into account the symmetries of the diffusion, the covariance and the regularity of its infinitesimal generator (Proposition 6), we end up with the following result:

Theorem 2. *Let $\mathcal{M} := (0, +\infty) \times_\alpha \mathbb{R}^3$ be a spatially flat Robertson-Walker space-time where the expansion function α has exponential growth. Let $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$ and let $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ be the relativistic diffusion starting from $(\xi_0, \dot{\xi}_0)$. Then, the invariant sigma field $Inv((\xi_s, \dot{\xi}_s)_{s\geq 0})$ of the whole diffusion coincides with the sigma field generated by the single variable $x_\infty \in \mathbb{R}^3$ up to $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$negligeable sets. Equivalently, if h is a bounded $\mathcal{L}-$harmonic function, there exists a bounded measurable function ψ on $\mathbb{R}^3$, such that*

$$h(\xi, \dot{\xi}) = \mathbb{E}_{(\xi,\dot{\xi})}[\psi(x_\infty)], \quad \forall (\xi, \dot{\xi}) \in T^1_+\mathcal{M}.$$

In other words, all the asymptotic information on the relativistic diffusion is encoded in the point $x_\infty \in \mathbb{R}^3$ or equivalently in the point $\xi_\infty = (\infty, x_\infty)$ of the causal boundary ∂M_c^+. The above theorem is thus very similar to Theorem 1 of [Bai08] asserting that the invariant sigma field of the relativistic diffusion in Minkowski space-time is generated by a random point on its causal boundary, which in that case identifies with the product $\mathbb{R}^+ \times \mathbb{S}^2$. It is thus tempting to ask if such a link between the Poisson and causal boundary holds in a more general context. Answering this question implies to determine the Poisson boundary of the relativistic diffusion in this more general setting, which unfortunately appears a hard task even in the case of general spatially flat Robertson-Walker space-time, where neither [Bai08]'s techniques, nor our present approach apply.

4 Proofs of the results

This last section is dedicated to the proofs of the different results stated above. Namely, the section 4.1 below is devoted to the proof of Proposition 1, and in Sections 4.2 and 4.3 we give the proofs of Theorems 1 and 2 respectively.

4.1 Existence, uniqueness, reduction of the dimension

We first give the proof of Proposition 1 concerning the existence, the uniqueness and the lifetime of the relativistic diffusion. The coefficients in the system of stochastic differential equations (2) being smooth, the first assertions follow from classical existence and uniqueness theorems, see for example Theorem (2.3) p. 173 of [IW89]. Next, the fact that the temporal process $(t_s, \dot{t}_s)$ is a sub-diffusion of the whole relativistic diffusion is an immediate consequence of Equation (2) and the pseudo-norm relation (3), which allows to express the norm of the spatial derivative $\dot{x}_s$ in term of the temporal process. Finally, the analogous result concerning the spherical sub-diffusion follows from a straightforward computation, namely setting $\Theta_s = (\Theta_s^1, \Theta_s^2, \Theta_s^3)$ where $\Theta_s^i := \dot{x}_s^i / |\dot{x}_s|$ to lighten the expressions, we have the following lemma:

Lemma 1. *The temporal process $(t_s, \dot{t}_s)$ and the spherical process $(t_s, \dot{t}_s, \Theta_s)$ are solutions of the following system of stochastic differential equations:*

$$\left\{ \begin{array}{l} dt_s = \dot{t}_s ds, \\ \\ d\dot{t}_s = d\dot{t}_s = -H(t_s)(\dot{t}_s^2 - 1)ds + \dfrac{3\sigma^2}{2}\dot{t}_s ds + dM_s^{\dot{t}}, \end{array} \right. \tag{4}$$

and

$$\left\{ \begin{array}{l} dt_s = \dot{t}_s ds, \\ \\ d\dot{t}_s = d\dot{t}_s = -H(t_s)(\dot{t}_s^2 - 1)ds + \dfrac{3\sigma^2}{2}\dot{t}_s ds + dM_s^{\dot{t}}, \\ \\ d\Theta_s^i = -\dfrac{\sigma^2}{\dot{t}_s^2 - 1} \times \Theta_s^i ds + dM_s^{\Theta^i}, \end{array} \right. \tag{5}$$

where the brakets of the martingales $M^{\dot{t}}$ and M^{Θ^i} are given by

$$\left\{ \begin{array}{l} d\langle M^{\dot{t}}, M^{\dot{t}} \rangle_s = \sigma^2 \left(\dot{t}_s^2 - 1\right) ds, \\ \\ d\langle M^{\dot{t}}, M^{\Theta^i} \rangle_s = 0, \\ \\ d\langle M^{\Theta^i}, M^{\Theta^j} \rangle_s = \dfrac{\sigma^2}{\dot{t}_s^2 - 1} \left(\delta_{ij} - \Theta_s^i \Theta_s^j\right) ds. \end{array} \right. \tag{6}$$

Remark 4. *From Remark 2, we know that, $\dot{t}_s \geq 1$ a.s. for all $s \geq 0$. In fact, the Hubble fuction $H = \alpha'/\alpha$ being non-increasing, using standard comparison techniques, it is easy to see that $\dot{t}_s > 1$ a.s for all $s > 0$, so that the term $\dot{t}_s^2 - 1$ in the denominators above never vanishes.*

Remark 5. *If we introduce the clock*

$$C_s := \sigma^2 \int_0^s \frac{du}{\dot{t}_u^2 - 1} du,$$

the process $\widetilde{\Theta}_s = (\widetilde{\Theta}_s^1, \widetilde{\Theta}_s^2, \widetilde{\Theta}_s^3)$ defined by $\widetilde{\Theta}_{C_s}^i := \Theta_s^i$ is nothing but a standard spherical Brownian motion on $\mathbb{S}^2 \subset \mathbb{R}^3$ and it is independent of the temporal sub-diffusion. In other words, the process $\Theta_s = (\Theta_s^1, \Theta_s^2, \Theta_s^3)$ is a time-changed spherical Brownian motion.

4.2 Asymptotic behavior of the diffusion

We now prove the results stated in Theorem 1 concerning the asymptotic behavior of the relativistic diffusion. We distinguish the cases of the temporal components of the diffusion (Proposition 2 below) and its spatial components (Proposition 3).

4.2.1 Asymptotic behavior of the temporal sub-diffusion

In this paragraph, the word "almost sure" refers to the law of the temporal sub-diffusion. The first point of Theorem 1 corresponds to the following proposition.

Proposition 2. *Let $(t_0, \dot{t}_0) \in (0, +\infty) \times [1, +\infty)$ and let $(t_s, \dot{t}_s)$ be the solution of Equation (4) starting from $(t_0, \dot{t}_0)$. Then, the process $\dot{t}_s$ is Harris-recurrent in $(1, +\infty)$ and if f is a monotone, $\nu_{H_\infty, \sigma}-$integrable function, or if it is bounded and continuous, one has the almost sure convergence:*

$$\lim_{s \to +\infty} \frac{1}{s} \int_0^s f(\dot{t}_u) du \overset{a.s.}{=} \nu_{H_\infty, \sigma}(f).$$

The proof of the proposition, which is given below, is based on standard comparison techniques and on the two following elementary lemmas. Recall that the Hubble function H is supposed to be non-increasing.

Lemma 2. *Given a constant $H > 0$, $\dot{t}_0 \in [1, +\infty)$ and a real standard Brownian motion B, the following stochastic differential equation*

$$d\dot{t}_s = -H \times \left(\dot{t}_s^2 - 1\right) ds + \frac{3\sigma^2}{2} \dot{t}_s ds + \sigma \sqrt{\dot{t}_s^2 - 1}\, dB_s$$

has a unique strong solution starting from $\dot{t}_0$, well defined for all times $s \geq 0$. Moreover, $\dot{t}_s$ admits the probability measure $\nu_{H,\sigma}$ introduced in Sect. 3.2 as an invariant measure. In particular, it is ergodic.

Lemma 3. *Let $(t_0, \dot{t}_0) \in (0, +\infty) \times [1, +\infty)$ and let $(t_s, \dot{t}_s)$ be the solution of Equation (4) starting from $(t_0, \dot{t}_0)$, where the martingale $M^{\dot{t}}$ is represented by a real standard Brownian motion B, i.e. $dM_s^{\dot{t}} = \sigma(\dot{t}_s^2 - 1)^{1/2} dB_s$. Let u_s and v_s be the unique strong solutions, well defined for all $s \geq 0$, and starting from $u_0 = v_0 = \dot{t}_0$, of the equations:*

$$du_s = -H(t_0) \left(u_s^2 - 1\right) ds + \frac{3\sigma^2}{2} u_s ds + \sigma \sqrt{u_s^2 - 1} dB_s,$$
$$dv_s = -H_\infty \left(v_s^2 - 1\right) ds + \frac{3\sigma^2}{2} v_s ds + \sigma \sqrt{v_s^2 - 1} dB_s.$$

Then, almost surely, for all $0 \leq s < +\infty$, one has $u_s \leq \dot{t}_s \leq v_s$.

Proof of Proposition 2. There exists a standard Brownian motion B such that the temporal process $(t_s, \dot{t}_s)$ is the solution of the stochastic differential equations

$$dt_s = \dot{t}_s ds, \quad d\dot{t}_s = -H(t_s) \times \left(\dot{t}_s^2 - 1\right) ds + \frac{3\sigma^2}{2} \dot{t}_s ds + \sigma\sqrt{\dot{t}_s^2 - 1}\, dB_s.$$

Let z_s be the unique strong solution, starting from $z_0 = \dot{t}_0$, of the stochastic differential equation

$$dz_s = -H_\infty \times \left(|z_s|^2 - 1\right) ds + \frac{3\sigma^2}{2} z_s ds + \sigma\sqrt{|z_s|^2 - 1}\, dB_s.$$

For $n \in \mathbb{N}$, let z_s^n be the process that coincides with $\dot{t}_s$ on $[0, n]$ and is the solution on $[n, +\infty)$ of the stochastic differential equation

$$dz_s^n = -H(t_0 + n) \times \left(|z_s^n|^2 - 1\right) ds + \frac{3\sigma^2}{2} z_s^n ds + \sigma\sqrt{|z_s^n|^2 - 1}\, dB_s.$$

By Lemma 3, for all $n \geq 0$ and $s \geq 0$, one has $z_s^n \leq \dot{t}_s \leq z_s$. By Lemma 2, both processes z_s^0 and z_s are ergodic in $(1, +\infty)$, in particular, they are Harris recurrent and so is $\dot{t}_s$. Now consider an increasing and $\nu_{H_\infty,\sigma}-$integrable function f, and fix an $\varepsilon > 0$. For all $n \in \mathbb{N}$, the function f is also integrable against the measure $\nu_{H(t_0+n),\sigma}$ and by dominated convergence theorem, when n goes to infinity, one has $\nu_{H(t_0+n),\sigma}(f) \longrightarrow \nu_{H_\infty,\sigma}(f)$. Choose n large enough so that we have $|\nu_{H(t_0+n),\sigma}(f) - \nu_{H_\infty,\sigma}(f)| \leq \varepsilon$. As $z_s^n \leq \dot{t}_s \leq z_s$ for $s \geq 0$, one has almost surely:

$$\int_0^s f(z_u^n) du \leq \int_0^s f(\dot{t}_u) du \leq \int_0^s f(z_u) du.$$

The integer n being fixed, by the ergodic theorem, we have that almost surely, as s goes to infinity:

$$\nu_{H_\infty,\sigma}(f) - \varepsilon \leq \nu_{H(t_0+n),\sigma}(f) \leq \liminf_{s \to +\infty} \frac{1}{s} \int_0^s f(\dot{t}_u) du,$$

and

$$\limsup_{s \to +\infty} \frac{1}{s} \int_0^s f(\dot{t}_u) du \leq \nu_{H_\infty,\sigma}(f).$$

Letting ε goes to zero, we conclude that almost surely, as s goes to infinity :

$$\frac{1}{s} \int_0^s f(\dot{t}_u) du \longrightarrow \nu_{H_\infty,\sigma}(f). \tag{7}$$

As any smooth function can be written as the difference of two monotone functions, the convergence (7) extends to functions in the set $C_b^1 = \{f,\ f' \text{ is bounded on } (1, +\infty)\}$, and then by regularization, to the set of bounded continuous functions on $(1, +\infty)$. □

4.2.2 Asymptotic behavior of the spatial components

The second and third points of Theorem 1 are the object of the next proposition:

Proposition 3. *Let $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$, and let $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ be the relativistic diffusion starting from $(\xi_0, \dot{\xi}_0)$. Then, as s goes to infinity, the spatial projection x_s converges almost surely to a random point $x_\infty \in \mathbb{R}^3$, and the process $\Theta_s = \dot{x}_s / |\dot{x}_s|$ is recurrent in $\mathbb{S}^2 \subset \mathbb{R}^3$.*

Proof. By the pseudo-norm relation (3), we have $|\dot{x}_s| = \sqrt{\dot{t}_s^2 - 1}/\alpha^2(t_s)$ for all $s \geq 0$. Therefore

$$|x_s - x_0| \leq \int_0^s |\dot{x}_u| du = \int_0^s \frac{\sqrt{\dot{t}_u^2 - 1}}{\alpha(t_u)} du \leq \int_0^s \frac{\dot{t}_u}{\alpha(t_u)} du = \int_{t_0}^{t_s} \frac{du}{\alpha(u)}.$$

By Remark 2, the process t_s goes almost surely to infinity with s. The increasing expansion function α having exponential growth, the last integral isthsu almost surely convergent, so that the total variation of x_s and the process itself are also convergent, whence the first point in the proposition. We have seen in Remark 5 that $\dot{x}_s/|\dot{x}_s| = \Theta_s = \widetilde{\Theta}_{C_s}$ is a time-changed spherical Brownian motion. By Proposition 2, the clock C_s goes almost surely to infinity with s, more precisely we have the almost sure convergence:

$$\lim_{s\to+\infty} \frac{C_s}{s} = \sigma^2 \lim_{s\to+\infty} \frac{1}{s}\int_0^s \frac{du}{\dot{t}_u^2 - 1} = \sigma^2 \int_1^{+\infty} \frac{\nu_{H_\infty,\sigma}(x)}{x^2-1} dx \in (0,+\infty).$$

In particular, the process Θ_s is recurrent in $\mathbb{S}^2$. □

4.3 Poisson boundary of the relativistic diffusion

The proof of Theorem 2 is divided into three parts. We first prove a Liouville theorem for the temporal sub-diffusion (Proposition 4), then we prove an analogous result for the spherical sub-diffusion (Proposition 5). Finally, we deduce the Poisson boundary of the global relativistic diffusion (Proposition 6).

4.3.1 A Liouville theorem for the temporal sub-diffusion

The infinitesimal generator of the temporal sub-diffusion $(t_s, \dot{t}_s)$, acting on smooth functions from $(0,+\infty)\times[1,+\infty)$ to $\mathbb{R}$, is given by

$$\mathcal{L}_H := \dot{t}\partial_t - H(t)(\dot{t}^2-1)\partial_{\dot{t}} + \sigma^2/2(\dot{t}^2-1)\partial_{\dot{t}}^2.$$

Following [CW00], we exhibit a shift coupling between two indepedent copies of the temporal diffusion to deduce that:

Proposition 4. *All bounded $\mathcal{L}_H-$harmonic functions are constant.*

Proof. The proof of Proposition 4 is based on the following fact: there is an automatic shift coupling between two independent solutions of the system (4). Consider B^1 and B^2 two independent standard Brownian motions defined on two measured spaces $(\Omega_1,\mathcal{F}_1)$ and $(\Omega_2,\mathcal{F}_2)$ as well as the processes $(t_s^1,\dot{t}_s^1)$ and $(t_s^2,\dot{t}_s^2)$, starting from $(t_0^1,\dot{t}_0^1) \neq (t_0^2,\dot{t}_0^2)$ (deterministic) and solution of the following systems, for $i=1,2$:

$$dt_s^i = \dot{t}_s^i ds, \;\; d\dot{t}_s^i = \left[-H(t_s^i)\left(|\dot{t}_s^i|^2-1\right) + \frac{3\sigma^2}{2}\dot{t}_s^i\right] ds + \sigma\sqrt{|\dot{t}_s^i|^2-1}dB_s^i.$$

Define $\tau_0 := \max(t_0^1, t_0^2)$. We denote by $\mathbb{P}_i$ the law of $(t_s^i,\dot{t}_s^i)$ and by $\mathbb{P} := \mathbb{P}_1 \otimes \mathbb{P}_2$ the law of the couple. From Remark 2, the processes t_s^i are strictly increasing. Denote by $(t^i)_s^{-1}$ their inverse, and define $u_s^i := \dot{t}^i[(t^i)_s^{-1}]$. Without loss of generality, one can suppose that $1 < u_{\tau_0}^1 < u_{\tau_0}^2$. By Itô's formula, for $s \geq \tau_0$, one has

$$\frac{1}{2}\log\left(\frac{|u_s^1|^2-1}{|u_s^2|^2-1}\right) = \frac{1}{2}\log\left(\frac{|u_{\tau_0}^1|^2-1}{|u_{\tau_0}^2|^2-1}\right) + Q_s + R_s + M_s, \tag{8}$$

where

$$Q_s := \sigma^2\left[(t^1)_s^{-1} - (t^2)_s^{-1}\right] - \sigma^2\left[(t^1)_{\tau_0}^{-1} - (t^2)_{\tau_0}^{-1}\right],$$

$$R_s := \frac{\sigma^2}{2}\left(\int_{\tau_0}^s \frac{u_r^2\left(|u_r^2|^2-1\right) - u_r^1\left(|u_r^1|^2-1\right)}{u_r^1\left(|u_r^1|^2-1\right)\times u_r^2\left(|u_r^2|^2-1\right)}dr\right),$$

and where M_s is a martingale whose bracket is given by:

$$\langle M \rangle_s = \sum_{i=1}^{2} \int_{(t^i)_{\tau_0}^{-1}}^{(t^i)_s^{-1}} \frac{|\dot{t}_u^i|^2}{|\dot{t}_u^i|^2 - 1} du \geq (t^1)_s^{-1} - (t^1)_{\tau_0}^{-1}. \tag{9}$$

Let us show that the coupling time $\tau_c := \inf\{s > \tau_0,\ u_s^1 = u_s^2\}$ is finite $\mathbb{P}-$almost surely. Consider the set $A := \{\omega \in \Omega_1 \times \Omega_2,\ \tau_c(\omega) = +\infty\}$. By definition, if $\omega \in A$ one has $u_s^1(\omega) < u_s^2(\omega)$ for $s > \tau_0$. We deduce that $R_s(\omega), Q_s(\omega) > 0$ for all $s > \tau_0$. Indeed, for $s > \tau_0$, one has :

$$\int_{\tau_0}^{s} \frac{dr}{u_r^1} > \int_{\tau_0}^{s} \frac{dr}{u_r^2}, \quad \text{and} \quad \int_{\tau_0}^{s} \frac{dr}{u_r^i} = \int_{\tau_0}^{s} \frac{dv}{\dot{t}^i[(t^1)_v^{-1}]} = (t^i)_s^{-1} - (t^i)_{\tau_0}^{-1}.$$

On the set A, by Equation (8), the martingale M_s thus admits the upper bound:

$$M_s + \frac{1}{2} \log\left(\frac{|u_{\tau_0}^1|^2 - 1}{|u_{\tau_0}^2|^2 - 1} \right) \leq \frac{1}{2} \log\left(\frac{|u_s^1|^2 - 1}{|u_s^2|^2 - 1} \right) \leq 0,$$

But by Equation (9), as $(t^1)_s^{-1}$ goes to infinity with s, we have also $\langle M \rangle_\infty = +\infty$ $\mathbb{P}-$almost surely. Therefore $\mathbb{P}(A) = 0$ and $\tau_c < +\infty$ $\mathbb{P}-$almost surely. In other words, $\mathbb{P}-$a.s. the two sets $(t^1_\cdot, \dot{t}^1_\cdot)_{\mathbb{R}^+}$ and $(t^2_\cdot, \dot{t}^2_\cdot)_{\mathbb{R}^+}$ intersect, where $(t^i_\cdot, \dot{t}^i_\cdot)_{\mathbb{R}^+}$ denotes the set of points of the curves $(t_s^i, \dot{t}_s^i)_{s \geq 0}$, $i = 1, 2$. Let us define the random times

$$T_1 := \inf\{s > 0, (t_s^1, \dot{t}_s^1) \in (t^2_\cdot, \dot{t}^2_\cdot)_{\mathbb{R}^+}\},$$
$$T_2 := \inf\{s > 0, (t_s^2, \dot{t}_s^2) \in (t^1_\cdot, \dot{t}^1_\cdot)_{\mathbb{R}^+}\}.$$

These variables are not stopping times for the filtration $\sigma((t_s^i, \dot{t}_s^i),\ i = 1, 2,\ s \leq t)_{t \geq 0}$, nevertheless they are finite $\mathbb{P}-$almost surely. Consequently, both sets $A_1 := \{\omega_1 \in \Omega_1,\ T_2 < +\infty\ \mathbb{P}_2 - \text{a.s.}\}$ and $A_2 := \{\omega_2 \in \Omega_2,\ T_1 < +\infty\ \mathbb{P}_1 - \text{a.s.}\}$ verify $\mathbb{P}_1(A_1) = \mathbb{P}_2(A_2) = 1$. Moreover, as the processes t_s^i are strictly increasing, one has

$$(t_{T_1}^1, \dot{t}_{T_1}^1) = (t_{T_2}^2, \dot{t}_{T_2}^2) \quad \mathbb{P} - \text{almost surely}. \tag{10}$$

Indeed, by definition of T_1 and T_2, there exists $u, v \in \mathbb{R}^+$ (random) such that $(t_{T_1}^1, \dot{t}_{T_1}^1) = (t_u^2, \dot{t}_u^2)$ and $(t_{T_2}^2, \dot{t}_{T_2}^2) = (t_v^1, \dot{t}_v^1)$. If $t_{T_1}^1 = t_u^2 < t_{T_2}^2$, as t_s^2 is strictly increasing, we would have $u < T_2$ and $(t_u^2, \dot{t}_u^2) \in (t^1_\cdot, \dot{t}^1_\cdot)_{\mathbb{R}^+}$ which would contradict the definition of T_2 as an infimum. Therefore, we have $t_{T_1}^1 \geq t_{T_2}^2$ and $t_{T_1}^1 = t_{T_2}^2$ by symmetry. Finally, using the monotonicity of t_s^i again, we conclude that $u = T_2$ and $v = T_1$, hence the coupling (10). Now let h be a bounded $\mathcal{L}_T-$harmonic function. Fix $\omega_2 \in \Omega_2$. The map $\omega_1 \in \Omega_1 \mapsto T_1(\omega_1, \omega_2)$ is a stopping time for the filtration $\sigma((t_s^1, \dot{t}_s^1),\ s \leq t)_{t \geq 0}$, and it is finite $\mathbb{P}_1-$almost surely. By the optional stopping theorem, one has

$$h(t_0^1, \dot{t}_0^1) = \mathbb{E}_1\left[h(t_{T_1}^1, \dot{t}_{T_1}^1)\right] = \int h(t_{T_1}^1, \dot{t}_{T_1}^1) d\mathbb{P}_1,$$

and integrating against $\mathbb{P}_2$, we get :

$$h(t_0^1, \dot{t}_0^1) = \int h(t_{T_1}^1, \dot{t}_{T_1}^1) d\mathbb{P}_1 \otimes d\mathbb{P}_2 = \int h(t_{T_1}^1, \dot{t}_{T_1}^1) d\mathbb{P}.$$

In the same way, we have

$$h(t_0^2, \dot{t}_0^2) = \int h(t_{T_2}^2, \dot{t}_{T_2}^2) d\mathbb{P}_1 \otimes d\mathbb{P}_2 = \int h(t_{T_2}^2, \dot{t}_{T_2}^2) d\mathbb{P}.$$

By (10), we conclude that $h(t_0^1, \dot{t}_0^1) = h(t_0^2, \dot{t}_0^2)$, *i.e.* the function h is constant. □

4.3.2 A Liouville theorem for the spherical sub-diffusion

We now extend the above Liouville theorem to the spherical sub-diffusion by using a second coupling argument, namely a mirror coupling argument on the sphere. To simplify the expressions in the sequel, we will denote by $(e_s)_{s\geq 0} := (t_s, \dot{t}_s, \Theta_s)_{s\geq 0}$ the spherical subdiffusion with values in the space $E := (0, +\infty) \times [1, +\infty) \times \mathbb{S}^2$ and by $\mathcal{L}_E$ its the infinitesimal generator acting on smooth functions from E to $\mathbb{R}$.

Proposition 5. *All bounded $\mathcal{L}_E-$harmonic functions are constant.*

Proof. Let us fix some initial conditions $e_0^1 = (t_0^1, \dot{t}_0^1, \Theta_0^1) \neq e_0^2 = (t_0^2, \dot{t}_0^2, \Theta_0^2)$ in E. As in the proof of the above Proposition 4, consider two independent solutions $(t_s^1, \dot{t}_s^1)$ and $(t_s^2, \dot{t}_s^2)$ of Equation (4), starting from $(t_0^1, \dot{t}_0^1)$ and $(t_0^2, \dot{t}_0^2)$ respectively, and such that there exists a shift coupling (10) in two different times T_1 and T_2 that are finite almost surely. We assume that both processes coincide after the coupling times, that is we suppose that $(t_{T_1+s}^1, \dot{t}_{T_1+s}^1) = (t_{T_2+s}^2, \dot{t}_{T_2+s}^2)$, for $s \geq 0$. Let us consider two independent spherical Brownian motions $\widetilde{\Theta}^i$ on $\mathbb{S}^2$, $i = 1, 2$, that are independent of the two above temporal diffusions and define for $s \geq 0$ and $i = 1, 2$:

$$\Theta_s^i := \widetilde{\Theta}^i \left(\int_0^s \frac{du}{|\dot{t}_u^i|^2 - 1} \right).$$

By Remark 5, the two diffusions $e_s^i := (t_s^i, \dot{t}_s^i, \Theta_s^i)$, $i = 1, 2$ are solutions of the stochastic differential equations (4–6), let us denote by $\mathbb{P}_i$ their law, define $\mathbb{P} := \mathbb{P}_1 \otimes \mathbb{P}_2$ and denote by $\mathbb{E}$ the associated expectation. Starting from this situation, it is easy to construct a coupling between the two paths e_s^i, $i = 1, 2$. Indeed, define a new process $({\Theta'}_s^2)_{s\geq 0}$, such that ${\Theta'}_s^2$ coincides with Θ_s^2 on the time interval $[0, T_2]$ and such that the future trajectory $({\Theta'}_s^2)_{s\geq T_2}$ is the reflection of $(\Theta_s^1)_{s\geq T_1}$ with respect to the median plan between the points $\Theta_{T^1}^1$ and $\Theta_{T^2}^2$, see figure 2 below.

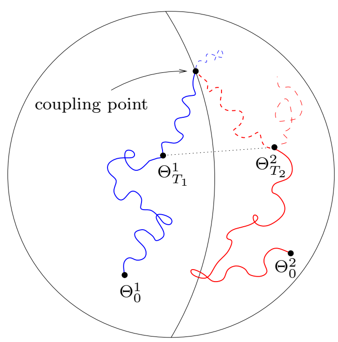

Figure 2: Mirror coupling of two independent spherical sub-diffusions.

The new process ${e'}_s^2 := (t_s^2, \dot{t}_s^2, {\Theta'}_s^2)$ is again a solution of Equations (4–6) and at the first time T^*, which is finite $\mathbb{P}-$almost surely, when the process $(\Theta_s^1)_{s\geq T_1}$ intersects the median big circle between $\Theta_{T^1}^1$ and $\Theta_{T^2}^2$, one has naturally:

$${e'}_{T_2+T^*}^2 = e_{T_1+T^*}^1$$

Now if h is a bounded $\mathcal{L}_E-$harmonic function, thanks to the above coupling and the optionnal stopping theorem, as in the proof of Proposition 4, we have $\mathbb{P}-$almost surely $h(e_0^2) = h(e_0^1)$ because

$$\mathbb{E}\left[\left(h({e'}_{T_2+T^*}^2) - h(e_{T_1+T^*}^1) \right) \right] = 0.$$

Therefore, the function h is constant, hence the result. ∎

4.3.3 Poisson boundary of the global relativistic diffusion

In order to describe the Poisson boundary of the whole relativistic diffusion $(\xi_s, \dot{\xi}_s)_{s\geq 0}$ starting from the one of the spherical subdiffusion, we need a few preliminaries. First notice that, thanks to the pseudo-norm relation (3), the invariant sigma field of the whole diffusion $(\xi_s, \dot{\xi}_s)_{s\geq 0}$ with values in $T^1_+\mathcal{M}$ coincides almost surely with the one of the diffusion process $(e_s, x_s)_{s\geq 0} = ((t_s, \dot{t}_s, \Theta_s), x_s)_{s\geq 0}$ with values in $E \times \mathbb{R}^3$ and whose infinitesimal generator $\mathcal{L}$ reads:

$$\mathcal{L} := \mathcal{L}_E + F(e)\,\partial_x, \quad \text{where} \;\; F(e) = F(t, \dot{t}, \Theta) := \Theta \times \frac{\sqrt{\dot{t}^2 - 1}}{\alpha(t)}. \tag{11}$$

Without loss of generality, we can suppose that the process $(e_s, x_s)_{s\geq 0}$ is defined on the canonical probability space $(\Omega, \mathcal{F})$ where $\Omega := C(\mathbb{R}^+, E \times \mathbb{R}^3)$ and $\mathcal{F}$ is the standard Borel sigma field. A generic $\omega \in \Omega$ writes $\omega = (\omega^1, \omega^2)$ where $\omega^1 = (\omega^1_s)_{s\geq 0} \in C(\mathbb{R}^+, E)$ and $\omega^2 = (\omega^2_s)_{s\geq 0} \in C(\mathbb{R}^+, \mathbb{R}^3)$. Without loss of generality again, we can suppose that $(e_s, x_s)_{s\geq 0}$ is the coordinate process, namely $(e_s, x_s) = (\omega^1_s, \omega^2_s)$ for all $s \geq 0$. Given (e, x) in $E \times \mathbb{R}^3$, we will denote by $\mathbb{P}_{(e,x)}$ the law of the process $(e_s, x_s)_{s\geq 0}$ starting from (e, x), and by $\mathbb{E}_{(e,x)}$ the associated expectation. Let us finally introduce the classical shift operators $(\theta_u)_{u\geq 0}$ acting on Ω and such that $\theta_u \omega = (\omega_{s+u})_{s\geq 0}$ for all $u \geq 0$. Recall that the tail sigma field $\mathcal{F}^\infty$ of the diffusion process $(e_s, x_s)_{s\geq 0}$ is defined as the intersection

$$\mathcal{F}^\infty := \bigcap_{s>0} \sigma((e_u, x_u), u > s),$$

and that the invariant sigma field $\mathrm{Inv}((e_s, x_s)_{s\geq 0})$ of $(e_s, x_s)_{s\geq 0}$ is the sub-sigma field of $\mathcal{F}^\infty$ composed of shift invariant events, namely events $A \in \mathcal{F}^\infty$ such that $\theta_u^{-1} A = A$ for all $u \geq 0$. In this setting, and starting from Proposition 4 and 5, Theorem 2 is equivalent to the following proposition:

Proposition 6. *Let h be a bounded $\mathcal{L}-$harmonic function on $E \times \mathbb{R}^3$. Then, there exists a bounded mesurable function ψ on $\mathbb{R}^3$ such that:*

$$h(e, x) = \mathbb{E}_{(e,x)}[\psi(x_\infty)], \quad \forall (e, x) \in E \times \mathbb{R}^3.$$

Equivalently, (e_0, x_0) being fixed, the invariant sigma field $Inv((e_s, x_s)_{s\geq 0})$ of the whole diffusion $(e_s, x_s)_{s\geq 0}$ starting from (e_0, x_0) coincides with $\sigma(x_\infty)$ up to $\mathbb{P}_{(e_0,x_0)}-$negligeable sets.

Proof. From the second point of Theorem 1, for all $(e, x) \in E \times \mathbb{R}^3$, the process $(x_s)_{s\geq 0}$ converges $\mathbb{P}_{(e,x)}-$almost surely to a random point $x_\infty = x_\infty(\omega) \in \mathbb{R}^3$. With a slight abuse of notation, let us still denote by x_∞ the random variable which coincides with x_∞ on the subset of Ω where the convergence occurs and which vanishes elsewhere. Thanks to the particular form (11) of the infinitesimal generator $\mathcal{L}$, let us remark the following facts:

1. for all starting points $(e, x) \in E \times \mathbb{R}^3$, the law of the process $(e_s, x + x_s)_{s\geq 0}$ under $\mathbb{P}_{(e,0)}$ coincide with the law of $(e_s, x_s)_{s\geq 0}$ under $\mathbb{P}_{(e,x)}$, in particular the law of the limit x_∞ under $\mathbb{P}_{(e,x)}$ is the law of $x + x_\infty$ under $\mathbb{P}_{(e,0)}$;
2. the push-forward measures of both measures $\mathbb{P}_{(e,0)}$ and $\mathbb{P}_{(e,x)}$ under the following mesurable map $\omega = (\omega^1, \omega^2) \mapsto (\omega^1, \omega^2 - x_\infty(\omega))$ coincide.

Let h be a bounded $\mathcal{L}-$harmonic function on $E \times \mathbb{R}^3$. From the classical duality between harmonic functions and invariant events, there exists a bounded $\mathrm{Inv}((e_s, x_s)_{s\geq 0})-$measurable random variable $Z : \Omega \to \mathbb{R}$, i.e. Z is $\mathcal{F}^\infty-$measurable and satisfies $Z(\theta_s \omega) = Z(\omega)$ for all $\omega \in \Omega$, such that $\forall\, (e, x) \in E \times \mathbb{R}^3$:

$$h(e, x) = \mathbb{E}_{(e,x)}[Z].$$

Moreover, $(e, x) \in E \times \mathbb{R}^3$ being fixed, for $\mathbb{P}_{(e,x)}-$almost all paths ω, we have:

$$Z(\omega) = \lim_{s\to+\infty} h(e_s(\omega), x_s(\omega)).$$

For $y \in \mathbb{R}^3$, consider the new random variable

$$Z^y(\omega) := Z((\omega^1, \omega^2 - x_\infty(\omega) + y)).$$

The variable Z^y is again $\mathrm{Inv}((e_s, x_s)_{s\geq 0})-$measurable. Indeed, since the constant function equal to y and the random variable Z are shift-invariant, for all $u \geq 0$ we have

$$Z((\omega^1_{\cdot+u}, \omega^2_{\cdot+u} - x_\infty(\omega_{\cdot+u}) + y)) = Z(\theta_u(\omega^1, \omega^2 - x_\infty(\omega) + y)) = Z((\omega^1, \omega^2 - x_\infty(\omega) + y)).$$

Since Z^y is a bounded $\mathrm{Inv}((e_s, x_s)_{s\geq 0})-$measurable variable, the function $(e, x) \mapsto \mathbb{E}_{(e,x)}[Z^y]$ is also a bounded $\mathcal{L}-$harmonic function. But from the point 2 of the beginning of the proof, for all starting points $(e, x, x') \in E \times \mathbb{R}^3$, we have

$$\mathbb{E}_{(e,x)}[Z^y] = \mathbb{E}_{(e,x')}[Z^y].$$

In other words, the harmonic function $(e, x) \mapsto \mathbb{E}_{(e,x)}[Z^y]$ is constant in x and its restriction to E is $\mathcal{L}_E-$harmonic. From Proposition 5, we deduce that the function $(e, x) \mapsto \mathbb{E}_{(e,x)}[Z^y]$ is constant. In the sequel, we will denote by $\psi(y)$ the value of this constant. Note that $y \mapsto \psi(y)$ is a bounded measurable function since $y \mapsto Z^y$ is. Let us now introduce an approximate unity $(\rho_n)_{n\geq 0}$ on $\mathbb{R}^3$, fix $\mathbf{x} \in \mathbb{R}^3$, $n \in \mathbb{N}$ and consider the "conditionned and regularized" version Z, namely:

$$Z^{\mathbf{x},n}(\omega) := \int_{\mathbb{R}^3} Z^y(\omega)\rho_n(\mathbf{x} - y)dy.$$

The exact same reasoning as above shows that $Z^{\mathbf{x},n}$ is a bounded $\mathrm{Inv}((e_s, x_s)_{s\geq 0})-$measurable variable so that the function $(e, x) \mapsto \mathbb{E}_{(e,x)}[Z^{\mathbf{x},n}]$ is constant. Hence, for all $\mathbf{x} \in \mathbb{R}^3$, $n \in \mathbb{N}$ and $(e, x) \in E \times \mathbb{R}^3$, there exists a set $\Omega^{\mathbf{x},n,(e,x)} \subset \Omega$ such that $\mathbb{P}_{(e,x)}(\Omega^{\mathbf{x},n,(e,x)}) = 1$ and such that for all paths ω in $\Omega^{\mathbf{x},n,(e,x)}$, we have:

$$Z^{\mathbf{x},n}(\omega) = \lim_{s\to\infty} \mathbb{E}_{(e_s(\omega),x_s(\omega))}[Z^{\mathbf{x},n}] = \mathbb{E}_{(e_0(\omega),x_0(\omega))}[Z^{\mathbf{x},n}] = \mathbb{E}_{(e,x)}[Z^{\mathbf{x},n}].$$

Let D be a countable dense set in $\mathbb{R}^3$ and consider the intersection

$$\Omega^{(e,x)} := \bigcap_{\mathbf{x}\in D, n\in\mathbb{N}} \Omega^{\mathbf{x},n,(e,x)}.$$

We have naturally $\mathbb{P}_{(e,x)}(\Omega^{(e,x)}) = 1$ and for $\omega \in \Omega^{(e,x)}$:

$$\forall \mathbf{x} \in D,\ n \in \mathbb{N}, \quad Z^{\mathbf{x},n}(\omega) = \mathbb{E}_{(e,x)}[Z^{\mathbf{x},n}].$$

Since the above expressions are continuous in $\mathbf{x}$, we deduce that the last inequality is true for all $\mathbf{x} \in \mathbb{R}^3$. In other words, we have shown that for all $\mathbf{x} \in \mathbb{R}^3$ and for all ω in $\Omega^{(e,x)}$:

$$Z^{\mathbf{x},n}(\omega) = \mathbb{E}_{(e,x)}[Z^{\mathbf{x},n}] = \int_{\mathbb{R}^3} \psi(y)\rho_n(\mathbf{x} - y)dy.$$

In particular, taking $\mathbf{x} = x_\infty(\omega)$, we obtain that for all $\omega \in \Omega^{(e,x)}$ and for all $n \in \mathbb{N}$:

$$Z^{x_\infty(\omega),n}(\omega) = \int_{\mathbb{R}^3} Z((\omega^1, \omega^2 + y))\rho_n(-y)dy = \int_{\mathbb{R}^3} \psi(y + x_\infty(\omega))\rho_n(y^{-1})dy.$$

Taking the integral in ω with respect to $\mathbb{P}_{(e,x)}$ on $\Omega^{(e,x)}$, we deduce that for all $n \in \mathbb{N}$:

$$\mathbb{E}_{(e,x)}\left[Z^{x_\infty,n}\right] = \int_{\mathbb{R}^3} \mathbb{E}_{(e,x)}[\psi(y + x_\infty)]\rho_n(-y)dy,$$

which, from the first point at the beginning of the proof yields

$$\int_{\mathbb{R}^3} h(e, x+y)\rho_n(-y)dy) = \int_{\mathbb{R}^3} \mathbb{E}_{(e,x+y)}[\psi(x_\infty)]\rho_n(-y)dy.$$

To conclude, recall that the infinitesimal generator of the diffusion is hypoelliptic so that $\mathcal{L}-$harmonic functions are continuous, hence we can let n go to infinity in the above expressions to get the desired result, namely:

$$h(e, x) = \mathbb{E}_{(e,x)}[\psi(x_\infty)].$$

□

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